ISRAEL JOURNAL OF MATHEMATICS, Vol. 73, No. 1, 1991

POLYHEDRAL EMBEDDINGS IN THE PROJECTIVE PLANE

BY

D. W. BARNETTE

Department of Mathematics, University of California at Davis, Davis, CA 95616, USA

ABSTRACT

We characterize the graphs that have polyhedral embeddings in the projective plane. We also prove that if one embedding of a graph is polyhedral then all embeddings of that graph are polyhedral.

1. Introduction

We shall say that an embedding of a graph G in a 2-manifold is a *polyhedral embedding* provided the faces are closed 2-cells, the vertices are at least 3-valent, and the faces meet the way faces in a convex polytope meet (i.e., two faces intersect on a vertex, an edge or not at all). A famous theorem of Steinitz [3] states that when the manifold is the 2-sphere, all such graphs are isomorphic to the graphs of vertices and edges of convex 3-polytopes. It also follows from Steinitz's theorem that a graph has a polyhedral embedding in the sphere if and only if it is planar and 3-connected. It is well known that each planar 3-connected graph has only one embedding in the sphere (see, for example, [4]).

In this paper we give necessary and sufficient conditions for a graph to have a polyhedral embedding in the projective plane.

Although a polyhedral map may have more than one embedding in the projective plane, we show that if a graph has a polyhedral embedding in the projective plane Π, then all embeddings in Π are polyhedral.

Received February 1, 1990

2. Definitions

All graphs in this paper are without loops on multiple edges. If a graph G is embedded in a 2-manifold M, then the *faces* of G are the closures of the connected components of $M - G$. It is easily seen that the interiors of faces of G are arcwise connected. The boundary of a face F will be denoted $\beta(F)$.

A graph G embedded in a 2-manifold is a *polyhedral map* provided each vertex is of valence at least 3, each face is a closed 2-cell, and every two faces have a connected intersection.

If G embedded in M is a polyhedral map, we also say that G has a *polyhedral embedding* in M.

We shall use two operations for constructing polyhedral maps. We shall say that a graph G_1 embedded in M is obtained from a graph G_2 embedded in M by *edge shrinking* provided shrinking an edge e of G_2 to a vertex v and coalescing multiple edges bounding any resulting 2-sided faces produces an embedding of G_1 in M. The inverse of shrinking edge e is called *splitting vertex v*.

If G is embedded in M and we add an edge e across a face F of M such that the endpoints of e do not lie on the same edge of F, we say that the resulting graph is obtained from G by *splitting face F*. Note that new vertices may or may not be introduced by this operation depending on whether the endpoints of e are vertices of G or lie in the relative interiors of edges of G.

By a theorem of the author [1], the polyhedral maps in the projective plane (called PPPM's) can be generated from a set of seven maps (see Fig. 1) by vertex splitting and face splitting. That is, if G is a PPPM then there is a sequence of PPPM's $G_0, G_1, \dots, G_n = G$ with G_0 one of the seven maps in Fig. 1 and each G_i obtained from G_{i-1} by either vertex splitting or edge splitting, for $1 \leq i \leq n$. The seven maps in Fig. 1 will be called the *minimal maps* for the projective plane.

If a simple closed curve C in a 2-manifold M bounds a cell that is a subset of M, we say that C is *planar*, otherwise we say that C is *nonplanar*.

3. Polyhedral embeddings in Π

THEOREM 1. *Let G be a 3-connected graph embedded in the projective plane Π and suppose that for every vertex v of G, $G - v$ is nonplanar. Then the embedding is polyhedral.*

PROOF. Let G be embedded in Π. Suppose some face F in G is not bounded by a simple closed curve. Since each vertex of G is at least 2-valent, each vertex of F will meet at least two edges of F. Thus F contains a simple circuit C.

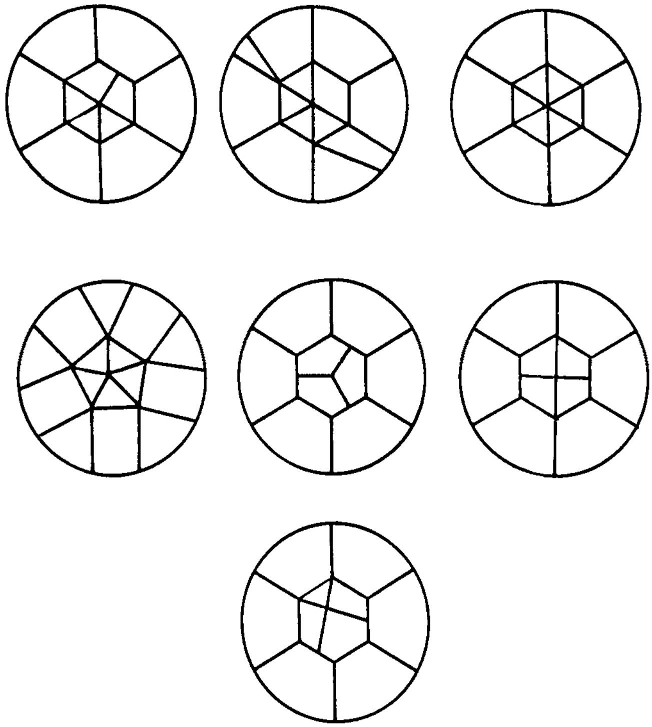

Fig. 1.

Case I. C is not the entire boundary of F and no other edge of $\beta(F)$ meets C. If C is planar, then either F is enclosed by C or F lies outside the cell X bounded by C. In the first case the connectedness of G implies that no edges of G lie inside C, thus F is the cell X and C is the entire boundary of F, a contradiction. In the second case, C and all vertices and edges inside it are separated from the rest of G contradicting the fact that G is connected. If C is nonplanar, then in a neighborhood of C is a nonplanar closed curve lying in F missing G. Cutting Π along C yields a cell with G embedded in it, a contradiction to our hypotheses.

Case II. C is not the entire boundary of F and an edge e of $\beta(F)$, that is not on C, meets C at a vertex v. Let e_1 and e_2 lie on C and meet at v. Let e_1, e_2, e be the clockwise cyclic ordering of these three edges about v. Let N be a neighborhood of v and suppose points of F lie in the portion of N clockwise between e_1 and e_2.

Since e is on $\beta(F)$ there is a point x of F in a portion of N either between e_2 and e or between e and e_1 (clockwise). Since F is arcwise connected, there is an arc connecting x to any point y in the portion of N between e_1 and e_2. Thus in F there is a closed curve C_1 lying in F meeting G only at v and containing x and y. If C_1 is planar, then it separates one of e_1 or e_2 from e which contradicts the 3-connectivity of G.

Suppose C_1 is nonplanar. We cut Π along C_1. This separates v into two vertices, v_1 and v_2, and produces a cell A containing the graph G' produced from G by separating v. Furthermore, v_1 and v_2 lie on the same face of the embedding of G' in A, thus we can identify v_1 and v_2 in A and obtain a planar embedding of G, contradicting our hypotheses.

We now have that C is the entire boundary of F and thus we may assume that every face of G is bounded by a simple closed curve.

Suppose now that our embedding of G is not polyhedral. Then there are two faces, F_1 and F_2, whose intersection is not connected.

We choose vertices x and y lying in different connected components of $F_1 \cap F_2$ and let P_i be a path in F_i meeting $\beta(F_i)$ only at x and y. Now, $P_1 \cup P_2$ is a simple closed curve. We treat two cases.

Case I. $P_1 \cup P_2$ is planar. In this case x and y separate G, a contradiction.

Case II. $P_1 \cup P_2$ is nonplanar. Consider $G - x$ (with the embedding in Π induced by the embedding of G). In this graph, $P_1 \cup P_2$ is a simple nonplanar closed curve meeting $G - x$ only at y. By the argument above, $G - x$ is planar, contradicting our hypotheses. Thus faces meet properly and the embedding of G is polyhedral.

THEOREM 2. *If G has a polyhedral embedding in the projective plane* Π, *then for every vertex x of G, $G - x$ is nonplanar.*

PROOF. By exhaustion, one can check this property for the seven minimal polyhedral maps for Π (Fig. 1). We now proceed by induction on the number of edges of G.

Let G be nonminimal.

Case I. G is obtained from a polyhedral map G_1 by adding an edge. If x is not a new vertex created by adding e, then since $G_1 - x$ is nonplanar, $G - x$ is nonplanar. If x is a new (and thus 3-valent) vertex of e, then $G - x$ is the same as G_1 minus one edge e_1. This, however, contains G_1 minus a vertex of e_1, which is nonplanar, thus $G - x$ is nonplanar.

Case II. G is obtained from a polyhedral map G_1 by splitting a vertex v of G_1 into two vertices v_1 and v_2. If x is not v_1 or v_2, then $G_1 - x$ is obtained from

$G - x$ by shrinking the edge $v_1 v_2$. Since edge shrinking preserves planarity and $G_1 - x$ is nonplanar, $G - x$ must be nonplanar.

If $x = v_1$ or v_2, then $G - x$ contains $G - \{v_1, v_2\} = G_1 - v$, which is nonplanar, thus $G - x$ is nonplanar.

THEOREM 3. *A graph G has a polyhedral embedding in* Π *if and only if it is embeddable in* Π, *3-connected, and for each vertex x,* $G - x$ *is nonplanar.*

PROOF. Sufficiency of these conditions is given by Theorem 1.

By a theorem of the author [2] the graph of every PPPM is 3-connected. Theorem 2 gives the necessity of the nonplanarity of $G - x$.

COROLLARY. *If G has a polyhedral embedding in* Π, *then every embedding in* Π *is polyhedral.*

PROOF. If G has a polyhedral embedding, then by Theorem 3 it satisfies the hypotheses of Theorem 1.

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